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ASYMPTOTIC ANALYSES OF THE BUCKLING OF IMPERFECT COLUMNS ON NONLINEAR ELASTIC FOUNDATIONS†

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Abstract—The buckling of a model imperfection-sensitive structure is studied, by a variety of techniques, for various kinds of deterministic and stochastic initial imperfections. The structure considered is an infinitely long column on a "softening" nonlinear elastic foundation. The techniques used are perturbation expansions (including the "two-time" variety), equivalent linearization and truncated hierarchy approximations; in all cases, asymptotic results are sought for small imperfection magnitudes, and the effects of the various kinds of imperfections are compared.

INTRODUCTION

IN THIS paper an imperfection-sensitive model structure will be studied by several techniques for various kinds of initial imperfections. The structure considered is an infinitely long uniform column resting on a "softening" nonlinear elastic foundation; the imperfections will be deterministic and stochastic initial displacements. Two types of deterministic initial displacements will be considered: purely harmonic deflections in the shape of the buckling mode of the perfect column, and localized dimples that decay exponentially. The stochastic initial displacements will be assumed to be stationary, random functions of position along the column, with multivariate Gaussian joint probability densities of the values at arbitrary collections of position coordinates. In all cases, the analyses will be aimed at finding results that are asymptotically valid for small imperfection magnitudes.

Earlier work on the same model with random imperfections is contained in [1], and the column with harmonic imperfections could be handled as a special case of Koiter's general theory [2]. The main purpose of the present paper, however, is to display and discuss a variety of techniques that can be brought to bear on problems of the types considered. These methods include: several kinds of perturbation expansions, equivalent linearization, "two-timing" expansions and truncated hierarchy approximations. An additional purpose is to provide a comparison of the relative effects of the different types of imperfections considered.

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DIFFERENTIAL EQUATION

The structure considered is an infinitely long column having bending stiffness EI, subjected to an axial load P, and restrained against lateral displacement $W(X)$ by a foundation that provides a restoring force-per-unit-length $k_1 W - k_3 W^3$. In the presence of an initial displacement $W_0(X)$, the governing differential equation is

$$EI\frac{d^4W}{dX^4}+P\left(\frac{d^2W}{dX^2}\right)+k_1W-k_3W^3 = -P\frac{d^2W_0}{dX^2}. \tag{1}$$

(Nonlinear terms containing *derivatives* of W were dropped from this equation.) With k_1 and k_3 assumed positive, the nondimensionalization

$$X = (EI/k_1)^{1/4}x$$
$$W = (k_1/k_3)^{1/2}w$$
$$P = 2\lambda(EIk_1)^{1/2}$$

and the introduction of the small imperfection parameter ε via

$$W_0 = \varepsilon(k_1/k_3)^{1/2}w_0$$

leads to

$$w'''' + 2\lambda w'' + w - w^3 = -2\lambda\varepsilon w_0'' \tag{2}$$

where $(\ )' \equiv d/dx(\ )$. Only bounded solutions of (2) will be sought.

The *linear* eigenvalue problem defined by setting $w_0 = 0$ and dropping the cubic term in (2) has the solutions

$$\begin{cases} w = \cos nx \\ \lambda = \dfrac{1}{2}\left(n^2+\dfrac{1}{n^2}\right) \end{cases} \tag{3}$$

where the origin of the x-coordinate is arbitrary. The *lowest* eigenvalue is $\lambda = 1$, corresponding to $n = 1$.

The solution of the nonlinear, nonhomogeneous equation (2) would provide a relation between the nondimensional load λ, the imperfection parameter ε and the nondimensional displacement w. The buckling load $\bar{\lambda}$ of the imperfect, nonlinear structure would then be defined as the maximum value reached by λ on that branch of the solution that emanates from $\lambda = 0$. This is illustrated by Fig. 1, which shows, qualitatively, the expected relations

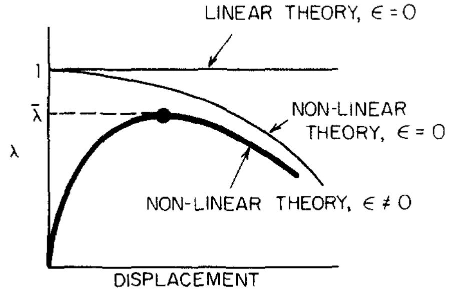

FIG. 1. Variation of λ with displacement.

between λ and an appropriate scalar measure of the displacement for the three cases (i) linear theory and $\varepsilon = 0$, (ii) nonlinear theory and $\varepsilon = 0$, and (iii) nonlinear theory and $\varepsilon \neq 0$.

HARMONIC IMPERFECTION

Perturbation expansion in displacement

With $w_0 = \cos x$, consider λ to be prescribed (between 0 and 1), and write

$$w = \eta \cos x + v(x) \tag{4}$$

where the average value of $v \cos x$ vanishes. (We can write this orthogonality requirement as $\langle v \cos x \rangle = 0$). Note that for any w, this prescription determines η and v unambiguously. We seek a relation between λ, ε and η; let us regard v and $\lambda\varepsilon$ as functions of η, expanded in the form

$$\left\{ \begin{aligned} v &= \sum_{m=2}^{\infty} \eta^m v_m(x) && (5) \\ \lambda\varepsilon &= \sum_{m=1}^{\infty} \eta^m B_m . && (6) \end{aligned} \right.$$

This approach is related, in spirit, to that of Koiter's general theory [2], as well as to Thompson's recent work [3] on discrete systems.

Substituting (5) and (6) into the differential equation (2), and equating to zero the coefficients of successive powers of η gives:

$$\begin{aligned} 0 &= 2B_1 \cos x - 2(1-\lambda) \cos x \\ L(v_2) &= 2B_2 \cos x \\ L(v_3) &= 2B_3 \cos x + \cos^3 x = 2B_3 \cos x + \tfrac{3}{4} \cos x + \tfrac{1}{4} \cos 3x \\ L(v_4) &= 2B_4 \cos x + 3v_2 \cos^2 x \\ L(v_5) &= 2B_5 \cos x + 3v_3 \cos^2 x + 3v_2^2 \cos x \end{aligned}$$

etc.

where

$$L(\ ) \equiv (\ )'''' + 2\lambda(\ )'' + (\ ) \tag{7}$$

Note that the prescribed orthogonality between v and $\cos x$ implies, via integration by parts, that $\langle L(v_n) \cos x \rangle = 0$; hence the coefficients of $\cos x$ in the right-hand side of these equations must vanish. If, in addition, we insist that v_n be analytic and bounded, we find that, for $\lambda < 1$,

$$B_n = v_n = 0 \quad \text{for } n \text{ even}$$

and

$$B_1 = 1 - \lambda, \qquad \left\{ \begin{aligned} B_3 &= -\tfrac{3}{8} \\ v_3 &= \frac{\cos 3x}{8(41 - 9\lambda)} \end{aligned} \right.$$

and so on. The relation (6) governing the loading parameter λ, the displacement measure η and the imperfection size ε, can be determined in this way as

$$\lambda\varepsilon = (1-\lambda)\eta - \frac{3}{8}\eta^3 - \frac{3}{64(41-9\lambda)}\eta^5 - \frac{3\eta^7}{64(41-9\lambda)^2} + \dots \tag{8}$$

Retention of just the first two terms on the right of (8), and maximizing λ with respect to η gives

$$(1-\bar{\lambda})^{3/2} = \frac{9\bar{\lambda}\varepsilon}{4\sqrt{2}} \tag{9}$$

as an approximate relation between the buckling load $\bar{\lambda}$ of the imperfect structure and the imperfection ε.

Asymptotically, for $\varepsilon \to 0$,

$$\bar{\lambda} \approx 1 - \left(\frac{9\varepsilon}{4\sqrt{2}}\right)^{2/3}. \tag{10}$$

The rigorous determination of additional terms in an asymptotic development of $\bar{\lambda}$ in powers of ε requires that more than two terms in the right-hand side of (8) be retained. Nevertheless, the evident smallness of these additional terms implies that the $\bar{\lambda}$–ε relation (9) found from just two terms is accurate over a larger range of ε than is the asymptotic result (10).

Perturbation expansion in a load parameter

A more traditional kind of perturbation procedure (used, for example, in [4]) can be pursued by letting $\lambda = 1-\delta^2/2$ and using δ as an expansion parameter. The differential equation becomes

$$w'''' + 2w'' + w - \delta^2 w'' - w^3 = -2\lambda\varepsilon w_0''(x) \tag{11}$$

with $w_0 = \cos x$. Now let

$$w = \alpha\delta\cos x + u$$

where $\langle u\cos x\rangle = 0$, and write

$$u = \sum_{n=1}^{\infty} \delta^n u_n(x) \tag{12}$$

$$\lambda\varepsilon = \sum_{n=1}^{\infty} \delta^n A_n. \tag{13}$$

We regard α as fixed, and, by substitution of (12) and (13) into (11) we get

$$0 = 2A_1\cos x$$

$$\mathscr{L}(u_2) = 2A_2\cos x$$

$$\mathscr{L}(u_3) = \left(2A_3 - \alpha + \frac{3\alpha^3}{4}\right)\cos x + \frac{\alpha^3}{4}\cos 3x$$

$$\mathscr{L}(u_4) = 2A_4\cos x + u_2'' + 3\alpha^2 u_2\cos^2 x$$

$$\mathscr{L}(u_5) = u_3'' + 3\alpha^2\cos^2 x\, u_3 + 3\alpha u_2^2\cos x + 2A_5\cos x$$

etc.

where

$$\mathscr{L}(\ ) = (\ )'''' + 2(\ )'' + (\ ). \tag{14}$$

Then, suppressing cos x in the right-hand sides *and* in u_n gives $A_n = u_n = 0$ for n even, and

$$A_1 = 0, \begin{cases} A_3 = \frac{1}{2}(\alpha - \frac{3}{4}\alpha^3) \\ u_3 = \dfrac{\alpha^2 \cos 3x}{256} \end{cases}$$

and so on. It can thus be shown that (12) gives

$$\lambda\varepsilon = \frac{1}{2}\left(\alpha - \frac{3}{4}\alpha^3\right)\delta^3 - \frac{3\alpha^5\delta^5}{2048} - \left[\frac{27\alpha^5 - 6\alpha^7}{2(256)^2}\right]\delta^7 + \dots.$$

If we reintroduce $\eta \equiv \alpha\delta$, this gives

$$\lambda\varepsilon = \frac{\delta^2}{2}\eta - \frac{3}{8}\eta^3 - \frac{3\eta^5}{2048}\left(1 - \frac{9\delta^2}{64}\right) - \frac{3\eta^7}{(256)^2} + \dots. \tag{15}$$

This result is not inconsistent with that given by (8). If, in equation (8), λ is replaced by $1 - \delta^2/2$, and the various terms on the right-hand side of (8) are expanded in powers of δ, equation (15) would evidently be obtained. It would appear, however, that (8) is inherently a more efficient expansion which, for a given number of terms, is more accurate than (15) over a large range of λ.

Equivalent linearization

This method (used in [1]) replaces the term w^3 in (2) by $\tau^2 w$; the solution of (2) for w is then

$$w = \eta \cos x$$

where

$$\eta = \frac{2\lambda\varepsilon}{2(1-\lambda) - \tau^2}. \tag{16}$$

The "best" value of τ is then assumed to be given by the requirement that $\langle w^4 \rangle = \tau^2 \langle w^2 \rangle$, which gives

$$\tfrac{3}{8}\eta^4 = \tfrac{1}{2}\tau^2\eta^2. \tag{17}$$

Eliminating τ^2 from (16) and (17) gives

$$\lambda\varepsilon = (1-\lambda)\eta - \tfrac{3}{8}\eta^3 \tag{18}$$

which, as far as it goes, agrees with the earlier results.

While the method of equivalent linearization thus gives an answer that is asymptotically exact, there is no obvious way to improve it systematically.

DIMPLE IMPERFECTION

Equivalent linearization

Now suppose that the imperfection shape $w_0(x)$ is continuously differentiable and satisfies the exponential-decay condition

$$|w_0(x)| < Me^{-\alpha|x|} \qquad (M, \alpha > 0). \tag{19}$$

Again, write $w^3 = \tau^2 w$ in the differential equation (11). Taking the Fourier transform of this equation, according to the definition

$$\tilde{f}(\omega) \equiv \int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx$$

gives

$$\tilde{w}(\omega) = \frac{2\lambda\varepsilon\omega^2\tilde{w}_0(\omega)}{(\omega^2-1)^2+\delta^2\omega^2-\tau^2}.$$

Then, taking the inverse transform gives

$$w(x) = \frac{\lambda\varepsilon}{\pi}\int_{-\infty}^{\infty}\frac{\omega^2\tilde{w}_0(\omega)e^{-i\omega x}\,d\omega}{(\omega^2-1)^2+\delta^2\omega^2-\tau^2}. \tag{20}$$

Let us assume, tentatively, $\tau < \delta$ for sufficiently small δ; it is shown in the Appendix that, under the hypothesis (19), an asymptotic approximation to $w(x)$ that is uniformly valid over the infinite range of x, for small δ, is given by

$$w(x) \approx \frac{\lambda\varepsilon}{2\sqrt{\delta^2-\tau^2}}e^{-|x|/2\sqrt{\delta^2-\tau^2}}[\tilde{w}_0(1)e^{-ix}+\tilde{w}(-1)e^{ix}]. \tag{21}$$

We will now impose the requirement

$$\int_{-\infty}^{\infty} w^4 dx = \tau^2\int_{-\infty}^{\infty} w^2\,dx \tag{22}$$

in order to get the "best" τ, and use (21) to calculate the required integrals.
Thus

$$\int_{-\infty}^{\infty} w^2\,dx = \frac{\lambda^2\varepsilon^2|\tilde{w}_0(1)|^2}{(\delta^2-\tau^2)^{3/2}}$$

and

$$\int_{-\infty}^{\infty} w^4\,dx = \frac{3}{8}\frac{\lambda^4\varepsilon^4|\tilde{w}_0(1)|^4}{(\delta^2-\tau^2)^{5/2}}$$

and so

$$\tau = \sqrt{\frac{3}{8}}\frac{\lambda\varepsilon|\tilde{w}_0(1)|}{(\delta^2-\tau^2)^{1/2}} \tag{23}$$

It is convenient to regard τ as a measure of the magnitude of the displacement. Thus, the λ, τ, ε relation given in (23) may be written as

$$(1-\lambda)\tau^2-\tfrac{1}{2}\tau^4 = \tfrac{3}{16}\lambda^2\varepsilon^2|\tilde{w}_0(1)|^2. \tag{24}$$

Maximizing λ with respect to τ then gives

$$\bar{\tau} = (1-\lambda)^{1/2} = \frac{\delta}{\sqrt{2}} \tag{25}$$

as the value of τ at buckling—and this verifies the initial assumption $\tau < \delta$. The buckling load $\bar{\lambda}$ then follows from (24) as

$$\bar{\lambda} = \frac{1}{1+\sqrt{\frac{3}{8}}\varepsilon|\tilde{w}_0(1)|}. \tag{26}$$

Asymptotically, for sufficiently small ε,

$$\bar{\lambda} \approx 1-\sqrt{\tfrac{3}{8}}\varepsilon|\tilde{w}_0(1)| \tag{27}$$

but the studies for harmonic imperfections permit us to hope that (26) may be accurate for a larger range of ε than (27). However, it must be admitted that this is no more than wishful thinking. In fact, there is no evident reason to conclude that even (27), obtained by the method of equivalent linearization, is asymptotically exact.

"Two-timing" perturbation expansion

The approximation (21) found for the deflection by the method of equivalent linearization suggests the possibility of a perturbation expansion in powers of δ that involves more than one length coordinate as independent variable. (The designation "two-timing" for such an approach stems from its use in initial value problems wherein time is the independent variable [5, 6]).

We introduce the variable $\zeta \equiv \delta x$, and consider w to be a function of both ζ and x. Then

$$\frac{dw}{dx} = \frac{\partial w}{\partial x}+\delta\frac{\partial w}{\partial \zeta} \tag{28}$$

$$\frac{d^2w}{dx^2} = \frac{\partial^2 w}{\partial x^2}+2\delta\frac{\partial^2 w}{\partial x\partial \zeta}+\delta^2\frac{\partial^2 w}{\partial \zeta^2} \tag{29}$$

$$\frac{d^3w}{dx^3} = \frac{\partial^3 w}{\partial x^3}+3\delta\frac{\partial^2 w}{\partial x^2\partial \zeta}+3\delta^2\frac{\partial^3 w}{\partial x\partial \zeta^2}+\delta^3\frac{\partial^3 w}{\partial \zeta^2} \tag{30}$$

and the differential equation (11) becomes

$$\left[\left(\frac{\partial^2}{\partial x^2}+2\delta\frac{\partial^2}{\partial x\partial \zeta}+\delta^2\frac{\partial^2}{\partial \zeta^2}+1\right)^2-\delta^2\left(\frac{\partial^2}{\partial x^2}+2\delta\frac{\partial^2}{\partial x\partial \zeta}+\delta^2\frac{\partial^2}{\partial \zeta^2}\right)\right]w-w^3 = -2\lambda\varepsilon w_0''(x). \tag{31}$$

Now we seek a bounded solution of the form

$$w = \delta u_1(x,\zeta)+\delta^2u_2(x,\zeta)+\delta^3u_3(x,\zeta)+\dots \tag{32}$$

that is consistent with

$$\lambda\varepsilon = A_1\delta + A_2\delta^2 + \ldots . \tag{33}$$

Guided, again, by the form of (21) we shall admit the possibility of discontinuities in the u_n's or their derivatives at $x = 0$ and $\zeta = 0$; but w, dw/dx, d^2w/dx^2 and d^3w/dx^3 *must* be continuous (since these correspond, physically, to deflection, slope, moment and shear, respectively). If these continuity conditions are written in terms of the expansion (32), and then asserted for the coefficient of δ^n in the resultant expression, it is found that the following combination of functions must be continuous in ζ and x:

$$u_n \tag{34}$$

$$\frac{\partial u_n}{\partial x} + \frac{\partial u_{n-1}}{\partial \zeta} \tag{35}$$

$$\frac{\partial^2 u_n}{\partial x^2} + 2\frac{\partial^2 u_{n-1}}{\partial x \partial \zeta} + \frac{\partial^2 u_{n-2}}{\partial \zeta^2} \tag{36}$$

$$\frac{\partial^3 u_n}{\partial x^3} + 3\frac{\partial^3 u_{n-1}}{\partial x^2 \partial \zeta} + 3\frac{\partial^3 u_{n-2}}{\partial x \partial \zeta^2} + \frac{\partial^3 u_{n-3}}{\partial \zeta^3} \qquad (n = 1, 2, 3, \ldots). \tag{37}$$

(It is understood here that $u_k \equiv 0$ for $k \leqq 0$.)

Substitution of (32) into (31), and insisting that the coefficient of δ^n vanish, provides the following three differential equations for $n = 1, 2, 3$:

$$\left(\frac{\partial^2}{\partial x^2} + 1\right)^2 u_1 = -2A_1 w_0''(x) \tag{38}$$

$$\left(\frac{\partial^2}{\partial x^2} + 1\right)^2 u_2 = -2A_2 w_0''(x) - 4\frac{\partial^2}{\partial x \partial \zeta}\left(\frac{\partial^2}{\partial x^2} + 1\right) u_1 \tag{39}$$

$$\left(\frac{\partial^2}{\partial x^2} + 1\right)^2 u_3 = -2A_3 w_0''(x) - 4\frac{\partial^2}{\partial x \partial \zeta}\left(\frac{\partial^2}{\partial x^2} + 1\right) u_2 - 2\frac{\partial^2}{\partial \zeta^2}\left(\frac{\partial^2}{\partial x^2} + 1\right) u_1 - 4\frac{\partial^4 u_1}{\partial x^2 \partial \zeta^2} + \frac{\partial^2 u_1}{\partial x^2} + u_1^3. \tag{40}$$

The real-valued solution of (38) is

$$u_1 = a_1(\zeta)e^{ix} + \bar{a}_1(\zeta)e^{-ix} + b_1(\zeta)xe^{ix} + \bar{b}_1(\zeta)xe^{-ix} + f_1(x)$$

where $(\bar{\ })$ means the complex conjugate of ( ), and f_1 (a function of x but not of ζ) is a bounded, real-valued, particular solution of (38). But the boundedness of w (for $\delta > 0$) implies $b_1 \equiv 0$. Thus

$$u_1 = a_1(\zeta)e^{ix} + \bar{a}_1(\zeta)e^{-ix} + f_1(x). \tag{41}$$

Similarly,

$$u_2 = a_2(\zeta)e^{ix} + \bar{a}_2(\zeta)e^{-ix} + f_2(x) \tag{42}$$

and

$$u_3 = a_3(\zeta)e^{ix} + \bar{a}_3(\zeta)e^{-ix} + f_3(x) + g_3(\zeta, x) \tag{43}$$

where

$$f_n''''+2f_n''+f_n = -2A_n w_0''(x) \qquad (n = 1, 2, 3) \tag{44}$$

and

$$\left(\frac{\partial^2}{\partial x^2}+1\right)^2 g_3 = \left(u_1^3+\frac{\partial^2 u_1}{\partial x^2}-4\frac{\partial^4 u_1}{\partial x^2 \partial \zeta^2}\right). \tag{45}$$

Let us stipulate that $f_n(x)$ have a bounded Fourier transform $\tilde{f}_n(\omega)$, but admit the possibility of jumps $[f_n(0)] \equiv f_n(0^+)-f_n(0^-)$, $[f_n'(0)] \equiv f_n'(0^+)-f_n'(0^-)$, etc. Then

$$\tilde{f}_n(\omega) = \frac{2A_n\omega^2\tilde{w}_0(\omega)+[f_n'''(0)]-i\omega[f_n''(0)]-(\omega^2-2)[f_n'(0)]+i\omega(\omega^2-2)[f_n(0)]}{(\omega^2-1)^2}. \tag{46}$$

The absence of a double pole at $\omega = 1$ requires that

$$[f_n'''(0)]-i[f_n''(0)]+[f_n'(0)]-i[f_n(0)] = -2A_n\tilde{w}_0(1) \tag{47}$$

$$-i[f_n''(0)]-2[f_n'(0)]+i[f_n(0)] = -A_n(4\tilde{w}_0(1)+2\tilde{w}_0'(1)). \tag{48}$$

These equalities would also prevent a double pole at $\omega = -1$. [Note that the analyticity of $\tilde{w}_0(\omega)$ in the neighborhood of $\omega = \pm 1$ is assured by the assumption (19) concerning the exponential decay of w_0.] Now, from the continuity requirements (34–37), we must have, for $n = 1$,

$$\begin{cases} [f_1(0)] = -[a_1(0)]-[\bar{a}_1(0)] \\ [f_1'(0)] = -i[a_1(0)]+i[\bar{a}_1(0)] \\ [f_1''(0)] = [a_1(0)]+[\bar{a}_1(0)] \\ [f_1'''(0)] = i[a_1(0)]-i[\bar{a}_1(0)]. \end{cases} \tag{49}$$

Hence—if we assume that $\tilde{w}_0(1) \neq 0$—the relations (47) imply that $A_1 = 0$, and equation (48) then gives $[a_1(0)] = 0$. Hence $f_1(x) = 0$, and

$$u_1 = a_1(\zeta)e^{ix}+\bar{a}_1(\zeta)e^{-ix} \tag{50}$$

where $a_1(\zeta)$ is continuous.

For $n = 2$, the continuity requirements (34–37) tell us that

$$\begin{aligned} [f_2(0)] &= -[a_2(0)]-[\bar{a}_2(0)] \\ [f_2'(0)] &= -i[a_2(0)]+i[\bar{a}_2(0)]-[a_1'(0)]-[\bar{a}_1'(0)] \\ [f_2''(0)] &= [a_2(0)]+[\bar{a}_2(0)]-2i[a_1'(0)]+2i[\bar{a}_1'(0)] \\ [f_2'''(0)] &= i[a_2(0)]-i[\bar{a}_2(0)]+3[a_1'(0)]+3[\bar{a}_1'(0)]. \end{aligned} \tag{51}$$

It then follows from equation (47) for $n = 2$ that

$$[a_1'(0)] = -\frac{A_2}{2}\tilde{w}_0(-1). \tag{52}$$

(Also, from (48), $[a_2(0)] = (i/2)iA_2[\tilde{w}_0(-1)+\tilde{w}_0'(-1)]$, but this will not be used.) Finally, we will use (45) to deduce some more information about $a_1(\zeta)$. The right-hand side of (45) is

$$a_1^3e^{3ix}+\bar{a}_1^3e^{-3ix}+[3a_1^2\bar{a}_1-a_1+4a_1'']e^{ix}+[3a_1\bar{a}_1^2-\bar{a}_1+4\bar{a}_1'']e^{-ix}$$

The vanishing of the "secular" terms in e^{ix} and e^{-ix} is necessary if g_3 is to be devoid of contributions like xe^{ix} and xe^{-ix} that, we have agreed, must not appear. Hence $a_1(\zeta)$ must satisfy the differential equation

$$4a_1'' - a_1 + 3a_1^2\bar{a}_1 = 0 \tag{53}$$

for $|\zeta| > 0$, together with the jump relation (52) at $\zeta = 0$, and a boundedness condition at infinity. Multiplying (53) by $\bar{a}_1'$, and adding the result to its conjugate, gives

$$4(a_1'\bar{a}_1')' - (a_1\bar{a}_1)' + \tfrac{3}{2}(a_1^2\bar{a}_1^2)' = 0. \tag{54}$$

Integrating from 0^+ to ∞, assuming $a_1(\infty) = a_1'(\infty) = 0$, gives

$$4a_1'(0^+)\bar{a}_1'(0^+) - |a_1(0)|^2 + \tfrac{3}{2}|a_1(0)|^4 = 0. \tag{55}$$

But note that if $a_1(\zeta)$ satisfies (53) in $(0^+, \infty)$, it is also the solution in $(0^-, -\infty)$, since $a_1(\zeta)$ is continuous. Hence,

$$a_1'(0^+) = -a_1'(0^-) = -\tfrac{1}{4}A_2\tilde{w}_0(-1) \tag{56}$$

with the use of (52). Thus

$$\frac{A_2^2}{4}|\tilde{w}_0(1)|^2 - |a_1(0)|^2 + \tfrac{3}{2}|a_1(0)|^4 = 0. \tag{57}$$

To sum up the state of approximation that has been attained for (32) and (33), we have

$$\begin{cases} w = \delta[a_1(\zeta)e^{ix} + \bar{a}_1(\zeta)e^{-ix}] & (58) \\ \lambda\varepsilon = \delta^2 A_2 & (59) \end{cases}$$

where $a_1(\zeta) = a_1(-\zeta)$ satisfies the differential equation (53) for $|\zeta| > 0$, with $a_1(0)$ and A_2 related by (57). If we introduce the displacement measure $\sigma \equiv \delta|a_1(0)|$, we have, from (57),

$$A_2^2 = \frac{4}{|\tilde{w}_0(1)|^2}\left[\frac{\sigma^2}{\delta^2} - \frac{3\sigma^4}{2\delta^4}\right]$$

and so

$$\lambda^2\varepsilon^2|\tilde{w}_0(1)|^2 = 4\sigma^2\delta^2 - 6\sigma^4$$

or

$$(1-\lambda)\sigma^2 - \tfrac{3}{4}\sigma^4 = \tfrac{1}{8}\lambda^2\varepsilon^2|\tilde{w}_0(1)|^2. \tag{60}$$

The critical value $\bar{\sigma}$ that maximizes λ in (60) is $\bar{\sigma} = \sqrt{\tfrac{2}{3}(1-\bar{\lambda})}$. Substituting this into (60) then gives the same answer (26) for $\bar{\lambda}$ as was found by the method of equivalent linearization.

The corroboration of this result by the two-timing approach certainly lends confidence in its validity, and even suggests that it may be asymptotically exact for sufficiently small ε; but it cannot be asserted that this has been established rigorously. In fact, a word of caution must be inserted concerning the possibility of continuing the present perturbation procedure to higher degrees of approximation. It may well be—indeed, it is likely—that the uniform validity of the expression (32) will break down unless more independently scaled space variables are introduced. Alternatively, it may be desirable (or necessary) to "stretch" one or both of the presently used space variables themselves, with the amount of stretch depending on δ. (See, for example, [6]). In any case, improving the presently found result by perturbation techniques is not a trivial job.

RANDOM IMPERFECTION

Equivalent linearization

Now suppose that $w_0(x)$ in equation (11) is a stationary, zero-mean random function, having a correlation function

$$R_{00}(\xi) = \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} w_0(x)w_0(x+\xi)\,dx \equiv \langle w_0(x)w_0(x+\xi)\rangle \tag{61}$$

and a power spectral density

$$S_{00}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{00}(\xi)e^{-i\omega\xi}\,d\xi. \tag{62}$$

Once again, write $w^3 = \tau^2 w$ in the differential equation; as in Ref. [1], we then find that the spectrum of the displacement w is

$$S_{11}(\omega) = \frac{4\lambda^2\varepsilon^2\omega^4 S_{00}(\omega)}{[(\omega^2-1)^2+\delta^2\omega^2-\tau^2]^2}.$$

The correlation function $R_{11}(\xi)$ of $w(x)$ is

$$\int_{-\infty}^{\infty} S_{11}(\omega)e^{i\omega\xi}\,d\omega.$$

The mean-square displacement

$$\Delta^2 = R_{11}(0) = \int_{-\infty}^{\infty} S_{11}(\omega)\,d\omega$$

is therefore

$$\Delta^2 = 4\lambda^2\varepsilon^2\int_{-\infty}^{\infty} \frac{\omega^4 S_{00}(\omega)\,d\omega}{[(\omega^2-1)^2+\delta^2\omega^2-\tau^2]^2}. \tag{63}$$

We assume that the imperfection $w_0(x)$ is *multivariate Gaussian*. (This means that the joint probability density $P[w_0^{(1)}, w_0^{(2)}, \ldots w_0^{(n)}]$ for the values $w_0^{(1)}$ at x_1, $w_0^{(2)}$ at x_2, etc. is given by

$$P = \frac{\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}w_0^{(i)}w_0^{(j)}\right\}}{(2\pi)^{n/2}(a)^{1/2}}$$

where

$$b_{ij} = \text{inverse of } a_{ij}$$

$$a_{ij} = R_{00}(x_i - x_j)$$

and $a = \det a_{ij}$.) Then, since linear functions of multivariate Gaussian quantities are also multivariate Gaussian, the displacements $w(x)$ *given by the linearized problem* are multivariate Gaussian. Hence

$$\langle w^4 \rangle \equiv \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} w^4 \, dx = \int_{-\infty}^{\infty} \frac{w^4 e^{-(w^2/2\Delta^2)}}{\Delta\sqrt{2\pi}} \, dw = 3\Delta^4.$$

Supposing, as before, that the "best" τ is given by

$$\langle w^4 \rangle = \tau^2 \langle w^2 \rangle = \tau^2 \Delta^2 \tag{64}$$

we get

$$\tau^2 = 3\Delta^2. \tag{65}$$

So far the analysis of [1] has been repeated; but now we seek an asymptotic result by assuming δ, $\Delta \ll 1$, and evaluate (63) asymptotically as follows:

$$\Delta^2 = 8\lambda^2\varepsilon^2 \int_0^\infty \frac{\omega^4 S_{00}(\omega)\, d\omega}{[(\omega^2-1)^2 + \delta^2\omega^2 - 3\Delta^2]^2}. \tag{66}$$

Let $\omega^2 = 1 + (\delta^2 - 3\Delta^2)^{1/2}\rho$; then

$$\Delta^2 \approx \frac{4\lambda^2\varepsilon^2}{(\delta^2-3\Delta^2)^{3/2}} \int_{-\infty}^{\infty} \frac{S_{00}(1)\, d\rho}{(\rho^2+1)^2} = \frac{2\pi\lambda^2\varepsilon^2 S_{00}(1)}{(\delta^2-3\Delta^2)^{3/2}}$$

and so

$$(1-\lambda)\Delta^{4/3} - \tfrac{3}{2}\Delta^{10/3} = \tfrac{1}{2}(2\pi)^{2/3}\lambda^{4/3}\varepsilon^{4/3}[S_{00}(1)]^{2/3}. \tag{67}$$

Maximizing λ with respect to Δ produces the result

$$(1-\bar{\lambda})^{5/4} = \tfrac{5}{2}(\tfrac{5}{6})^{1/4}[\pi S_{00}(1)]^{1/2}\bar{\lambda}\varepsilon. \tag{68}$$

For sufficiently small ε, this gives

$$\bar{\lambda} \approx 1 - \tfrac{5}{2}(3)^{-1/5}[\pi S_{00}(1)]^{2/5}\varepsilon^{4/5} \tag{69}$$

but we may hope that (68) gives a relation between $\bar{\lambda}$ and ε that is better for a larger range of ε.

Other methods

It would be desirable to try to check the result for random imperfections by one or more independent approaches. But while perturbation methods were available to us in the case of harmonic imperfections and, with two-timing, for dimples, we have been unable to concoct a perturbation scheme for the case of random imperfections.

Another approach that has been widely used in stochastic problems is that based on the so-called truncated–hierarchy idea. In this method (which is not well codified in any unique way) a differential equation for the correlation function of the unknown (w in the present case) is formulated; but it is then found (in all but the simplest problems) that such an equation introduces one or more other kinds of correlation function. Formulating equations for these new unknowns introduces still others—and, typically, one can never catch up. To end the process, one truncates this hierarchy of equations at some suitable stage, and introduces plausible approximations for enough unknown functions to render the system determinate.

In the presently considered nonlinear problem, the method does not look very attractive since the number of equations needed to improve the most primitive possible approximation appears to escalate very rapidly. It may, nevertheless, be of interest to show how the simplest approximation works.

Let

$$R_{11}(\xi) = \langle w(x)w(x+\xi)\rangle = R_{11}(-\xi)$$

$$R_{10}(\xi) \equiv \langle w(x)w_0(x+\xi)\rangle$$

$$R_{01}(\xi) = \langle w_0(x)w(x+\xi)\rangle = R_{10}(-\xi)$$

$$R_{00}(\xi) \equiv \langle w_0(x)w_0(x+\xi)\rangle = R_{00}(-\xi).$$

The governing differential equation can be written

$$\frac{\partial^4 w}{\partial \xi^4}(x+\xi)+2\lambda\frac{\partial^2 w}{\partial \xi^2}(x+\xi)+w(x+\xi)-w^3(x+\xi) = -2\lambda\varepsilon\frac{\partial^2 w_0}{\partial \xi^2}(x+\xi). \tag{70}$$

Multiplying by $w(x)$ and averaging over the infinite beam gives

$$R''''_{11}(\xi)+2\lambda R''_{11}(\xi)+R_{11}(\xi)-\langle w(x)w^3(x+\xi)\rangle = -2\lambda\varepsilon R''_{10}(\xi). \tag{71}$$

Similarly, replacing ξ by $-\xi$ in (70), multiplying by $w_0(x)$, and averaging over x gives

$$R''''_{10}(\xi)+2\lambda R''_{10}(\xi)+R_{10}(\xi)-\langle w_0(x)w^3(x-\xi)\rangle = -2\lambda\varepsilon R''_{00}(\xi). \tag{72}$$

We can stop here, and solve (71) and (72) for $R_{11}(\xi)$ and $R_{10}(\xi)$ by making suitable approximations for $\langle w(x)w^3(x+\xi)\rangle$ and $\langle w_0(x)w^3(x-\xi)\rangle$. Let us do this by supposing that the joint probability densities $P[w_0(x), w(x+\xi)]$ and $P[w(x), w(x+\xi)]$ are both Gaussian. Then [7]

$$\langle w(x)w^3(x+\xi)\rangle = 3\Delta^2 R_{11}(\xi)$$

$$\langle w_0(x)w^3(x-\xi)\rangle = 3\Delta^2 R_{01}(-\xi) = 3\Delta^2 R_{10}(\xi).$$

Hence, (71) and (72) become

$$\mathscr{L}\{R_{11}(\xi)\}-\delta^2 R''_{11}(\xi)-3\Delta^2 R_{11}(\xi) = -2\lambda\varepsilon R''_{10}(\xi) \tag{73}$$

$$\mathscr{L}\{R_{10}(\xi)\}-\delta^2 R''_{10}(\xi)-3\Delta^2 R_{10}(\xi) = -2\lambda\varepsilon R''_{00}(\xi). \tag{74}$$

Solving these equations by taking Fourier transforms [and recalling the relation (62)] gives

$$S_{11}(\omega) = \frac{4\lambda^2\varepsilon^2\omega^4 S_{00}(\omega)}{[(\omega^2-1)^2+\delta^2\omega^2-3\Delta^2]^2} \tag{75}$$

and then calculating $\Delta^2 = 2\int_0^\infty S_{11}(\omega)\,d\omega$ returns the relation (66) found earlier, by the method of equivalent linearization. Thus, the same asymptotic results (68) and (69) also follow.

To carry this process one logical step further would evidently require the introduction of higher order correlations of the type

$$R_{ijkl}(\xi,\eta,\zeta) = \langle w_i(x)w_j(x+\xi)w_k(x+\eta)w_l(x+\zeta)\rangle \qquad (i,j,k,l = 0,1)$$

(where $w_1 \equiv w$) together with many more differential equations governing them, as well as reasonable assumptions to approximate averages like

$$\langle w_i(x)w_j(x+\xi)w_k(x+\eta)w_j^3(x+\zeta)\rangle \qquad (i,j,k,l = 0,1)$$

in terms of R_{ijkl} and R_{ij}. We have not attempted this extension, and do not particularly recommend it.

COMPARISON OF RESULTS

The asymptotic results found for various kinds of imperfections will be recapitulated and compared.

In each case, the imperfection shape has the form

$$W = \varepsilon(k_1/k_3)^{1/2} w_0(x)$$

in terms of the nondimensional coordinate $x = (k_1/EI)^{1/4}X$. The buckling load is $\bar{P} = 2\bar{\lambda}(EIk_1)^{1/2}$, and the relations between $\bar{\lambda}$ and ε are as follows for sufficiently small ε:

(i) $w_0 = \cos x$:

$$\left.\begin{aligned} (1-\bar{\lambda})^{3/2} &\approx \frac{9\bar{\lambda}\varepsilon}{4\sqrt{2}} \\ \bar{\lambda} &\approx 1-\left(\frac{9}{4\sqrt{2}}\right)^{2/3}\varepsilon^{2/3} \end{aligned}\right\} \tag{76}$$

(ii) $|w_0| < M\mathrm{e}^{-\alpha|x|}, M > 0, \alpha > 0$:

$$\left.\begin{aligned} \bar{\lambda} &\approx \frac{1}{1+\sqrt{\frac{3}{8}}|\tilde{w}_0(1)|\varepsilon} \\ \bar{\lambda} &\approx 1-\sqrt{\tfrac{3}{8}}|\tilde{w}_0(1)|\varepsilon \end{aligned}\right\} \tag{77}$$

where

$$\tilde{w}_0(1) = \int_{-\infty}^{\infty} w(x)\mathrm{e}^{ix}\,\mathrm{d}x$$

(iii) $w_0(x)$ random, stationary, Gaussian:

$$\left.\begin{aligned} (1-\bar{\lambda})^{5/4} &\approx \tfrac{5}{2}(\tfrac{5}{6})^{1/4}[\pi S_{00}(1)]^{1/2}\bar{\lambda}\varepsilon \\ \bar{\lambda} &\approx 1-\tfrac{5}{2}(3)^{-1/5}[\pi S_{00}(1)]^{2/5}\varepsilon^{4/5} \end{aligned}\right\} \tag{78}$$

where

$$S_{00}(1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{00}(\xi)\mathrm{e}^{-i\xi}\,\mathrm{d}\xi$$

and

$$R_{00}(\xi) = \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L} w_0(x)w_0(x+\xi)\,\mathrm{d}x$$

We expect that the first relation in each of equations (76–78) is more accurate than the second, but with decreasing confidence in this conclusion for cases (i)–(iii). While we are reasonably certain that (76) is asymptotically exact, we are somewhat less sure for case (ii), and really not at all convinced that the results for the random imperfection are asymptotically exact. Finally, we note that the degradations of buckling strength, as measured by their dependence on the degree of ε, are greatest for the harmonic imperfection, less for the random imperfection, and least for the dimple.

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APPENDIX

Asymptotic evaluation of an integral

Suppose $|w_0(x)| < Me^{-\alpha|x|}$, $M > 0$, $\alpha > 0$, and let

$$\tilde{w}_0(\omega) = \int_{-\infty}^{\infty} w(x) e^{i\omega x}\,dx$$

We wish to find an asymptotic expression for

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega^2 \tilde{w}_0(\omega) e^{-i\omega x}\,d\omega}{(\omega^2-1)^2+\delta^2\omega^2-\tau^2} \qquad (\tau < \delta)$$

that is uniformly valid for all x for sufficiently small δ.

Consider first $x > 0$; since $w_0(\omega)$ is clearly analytic for $|Im(\omega)| < \alpha$, the integral for $g(x)$ can be shifted in the complex ω plane to provide the relation

$$g(x) = \frac{1}{2\pi}\int_{-\infty-i\alpha_1}^{-\infty-i\alpha_1} \frac{\omega^2 \tilde{w}_0(\omega) e^{-i\omega x}\,d\omega}{(\omega^2-1)^2+\delta^2\omega^2-\tau^2} - i\,[\text{Residues at } \omega_1, \omega_2]$$

where $\omega_{1,2} \approx \pm 1 - (i/2)\sqrt{\delta^2-\tau^2} + 0(\delta^2)$ are the poles of the integrand in the lower half plane, and $\frac{1}{2}\sqrt{\delta^2-\tau^2} < \alpha_1 < \alpha$. With α_1 held fixed, the integral is bounded by $Ke^{-\alpha_1|x|}$ $(K > 0)$; then, evaluating the residues yields

$$g(x) = \frac{1}{4(\delta^2-\tau^2)^{1/2}}[\tilde{w}_0(1)+0(\delta)]e^{-\{i+[(\delta^2-\tau^2)^{1/2}/2]+0(\delta^2)\}x}$$
$$+[\tilde{w}_0(-1)+0(\delta)]e^{\{i-[(\delta^2-\tau^2)^{1/2}/2]+0(\delta^2)\}x} + 0)Ke^{-\alpha_1 x})$$

It follows that, for $x > 0$,

$$g(x) \approx \frac{1}{4(\delta^2-\tau^2)^{1/2}}[\tilde{w}_0(1)e^{-ix} + \tilde{w}_0(-1)e^{ix}]e^{-[(\delta^2-\tau^2)^{1/2}/2]x}$$

uniformly for x as $\delta \to 0$. A similar calculation for $x < 0$ gives the combined result for all x

$$g(x) \approx \frac{1}{4(\delta^2-\tau^2)^{1/2}}[\tilde{w}_0(1)e^{-ix}+\tilde{w}_0(-1)e^{ix}]e^{-[(\delta^2-\tau^2)^{1/2}/2]|x|}$$

from which the expression (21) for the integral (20) was written.

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Абстракт—Исследуется разными методами потеря устойчивости модели конструкции чувствительной к неправильностям для разных случаев детерминистических и стохастических начальных непра вильностей Рассматриваемая конструкция представляет собой бесконечную длинную колонну на "размягченном" нелинейном упругом основании. Используются методы разложения по возму щениям /заключая "двувременное' множество/, эквивалентной линеаризации и отбрасывания по иерархии членов в приближениях. Во всех случаях ищется асимптотические результаты, для малых значений неправильностей и сравниваются эффекты для разных их классов.